Dye's theorem in the almost continuous category

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Abstract

Suppose X and Y are Polish spaces with non-atomic Borel probability measures μ and ν and suppose that T and S are ergodic measure-preserving homeomorphisms of (X, μ) and (Y, ν). Then there are invariant G_δ subsets $X' \subset X$ and $Y' \subset Y$ of full measure and a homeomorphism $\varphi : X' \to Y'$ which maps $\mu|_{X'}$ to $\nu|_{Y'}$ and maps T-orbits onto S-orbits. We also deal with the case where T and S preserve infinite invariant measures.

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Section 1. Introduction

Suppose $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ are measure-preserving dynamical systems on probability spaces. An orbit equivalence between X and Y is a measurable map $\varphi : X' \to Y'$, where X' and Y' are invariant subsets of measure one, with a measurable inverse $\psi : Y' \to X'$ such that $\varphi^* \mu = \nu$ and φ maps T-orbits onto S-orbits. A fundamental theorem of Dye [D1] asserts that any two measure-preserving systems are orbit equivalent.

Recently Hamachi and Keane [HK] proved that the binary and ternary odometers are actually orbit equivalent in a stronger sense, namely the orbit equivalence φ can be chosen to be continuous with a continuous inverse when restricted to an appropriate invariant subset of measure 1. Such an orbit equivalence φ is called almost continuous or finitary, the term finitary being more appropriate in the context of maps defined on sequence spaces, such as the odometers. We will use the term almost continuous as we will be working in a more general setting. There have been a number of results since then asserting the existence of an almost continuous orbit equivalence for various classes of ergodic measure-preserving homeomorphisms or maps which are homeomorphisms after restriction to an invariant subset of full measure, as is the case with odometers. See [HKR], [R1], [R2] and [RR3]. Perhaps the most general of these is the result of Hamachi, Keane and Roychowdhury [HKR] asserting that any two adic transformations are almost continuously orbit equivalent. We mention here also the celebrated work of Keane and Smorodinsky [KS1], [KS2] on finitary isomorphism, the paper of del Junco [J] on finitary unilateral isomorphism and the recent papers of Roychowdhury and Rudolph [RR1], [RR2] on finitary Kakutani equivalence.

Here we will prove the following general result, an almost continuous version of Dye's theorem, which includes all the orbit equivalence results mentioned above and much more. Recall that a subset of a Polish space is Polish if and only if it is a G_δ.

Theorem 1. *Suppose X and Y are Polish spaces with non-atomic Borel probability measures μ and ν and suppose that T and S are ergodic measure-preserving*

homeomorphisms of (X, μ) and (Y, ν). Then there are invariant G_δ subsets $X' \subset X$ and $Y' \subset Y$ of full measure and a homeomorphism $\varphi : X' \to Y'$ which maps $\mu|_{X'}$ to $\nu|_{Y'}$ and maps T-orbits onto S-orbits.

We will also prove the analogous result in the case where the invariant measure is infinite, Theorem 3 in Section 3 below. We thank Sasha Danilenko for an insight which simplified the proof of Theorem 3.

Theorem 1 says that the restrictions of T and S to X' and Y' are topologically orbit equivalent, in the sense of Giordano, Putnam and Skau [GPS], via a map which also carries μ to ν. Our proof is a combination of the techniques used in well-known proofs of Dye's theorem (see for example [KW] or [HIK]) with the Hamachi-Keane technique. For those familiar with [HK] we remark that we use constructs very similar to theirs but in a different setting with different terminology.

Of course, Theorem 1 also applies to discontinuous T and S as well, provided they have restrictions to invariant dense G_δ subsets of full measure which are continuous. This class includes odometers, interval exchange maps and adic transformations among others.

We remark that if μ and ν have full supports then X' and Y' in Theorem 1 are necessarily dense G_δ's. Theorem 1 is both measure-theoretic and topological in nature. By the above remark the purely topological character is that of generic orbit equivalence as defined by Sullivan, Weiss and Wright, [SWWr]. They prove a very general result, namely any two discrete groups of homeomorphisms of Polish spaces are continuously orbit equivalent after restriction to invariant dense G_δ subsets. In the special case of single homeomorphisms which possess some invariant probability of full support their result follows from ours.

Before we proceed with the proof of Theorem 1 we mention some questions which arise from this work. First, it is likely that our methods will show that any ergodic action of a discrete amenable group G by homeomorphisms of a Polish space preserving a probability measure is orbit equivalent to such an action with group $G = Z$. Is there an almost continuous analogue to the theorem of Dye [D2] which states that any isomorphism of the full groups of two countable discrete groups of transformations preserving a probability measure is implemented by an orbit

3

equivalence? Is there a theory in the case of non-singular measures with some, or all the features of the measurable theory (see [K], [HO], [KW])? Is there anything that distinguishes the almost continuous classification from the measurable classification? In other works might it be the case that if two non-singular homeomorphisms (or more generally groups of homeomorphisms) are measurably orbit equivalent then they must also be almost continuously orbit equivalent?

Returning to integer actions, note that every orbit equivalence between T and S is also an isomorphism between T and S' where S' is a map with the same orbits as S so that $S'y = S^{n(y)}y$ and $S(y) = S'^{m(y)}y$ where n and m are integer valued functions (co-cycles) on Y. Can we require in Theorem 1 that m and n be continuous on the set Y'? This question is motivated by the definition due to Giordano, Putnam and Skau [GPS] of strong topological orbit equivalence.

Section 2. Proof of Theorem 1

To prove Theorem 1 we first reduce it to a special case. Let us say a topological space X is **fractured** if its topology has a countable base consisting of clopen subsets. (We do not know if any zero-dimensional Polish space is fractured.) In a fractured space every open set is a union of countably many disjoint clopen sets. Note that any subset of a fractured space is trivially fractured.

Theorem 2. *Theorem 1 holds with the additional hypothesis that X and Y are fractured.*

How does Theorem 1 follow from Theorem 2? It suffices to show that X has an invariant G_δ subset X' of measure 1 which is fractured, since X' is again Polish. To construct X' first observe that for a fixed $x \in X$ at most countably many of the metric spheres $S(x, r) = \{y : d(x, y) = r\}$ can have positive measure. This means that we may find a countable dense subset $R_x \subset \mathbb{R}^+$ such that the spheres $S(x, r), r \in R_x$, all have measure zero. Let $\{x_i : i \in I\}$ be a countable dense subset of X and set

$$X^\sharp = (\bigcup_{i \in I} \bigcup_{r \in R_{x_i}} S(x_i, r))^c.$$

4

$X^\sharp$ is a G_δ of measure one and it is easy to see that $X^\sharp$ is fractured, so $X' = \bigcap_{n \in \mathbb{Z}} T^n X^\sharp$ is the desired invariant G_δ of measure one.

The rest of this section is devoted to the proof of Theorem 2. We will implicitly use fixed complete metrics on X and Y. We will state definitions and lemmas for (X, μ, T) with the understanding that we will use them for (Y, ν, S) as well. Since the support of μ is an invariant G_δ subset of full measure we can and shall assume henceforth that the measures μ and ν have full support. Thus any non-empty open set has non-zero measure. The fact that X is fractured and μ is non-atomic implies that every non-empty open set contains non-empty clopen sets of measure as small as we please. We will frequently use (implicitly) the fact that the clopen sets form an algebra invariant under T so that all finite set operations involving clopen sets and powers of T yield clopen sets. If A is a measurable subset of X and $A_1, A_2, \ldots$ are subsets of A we will say that $\{A_i\}$ **fills out** A if $\mu(\bigcup_i A_i) = \mu(A)$.

Lemma 1. *Suppose A and B are open subsets of X such that $\mu(A) = \mu(B)$. Then there are disjoint clopen sets $A_1, A_2, \ldots \subset A$ which fill out A and integers $n_1, n_2, \ldots$ such that the sets $B_i := T^{n_i} A_i$ are disjoint and contained in B (and fill out B).*

Proof: Suppose A_i and B_i have been defined for all $i < m$. Set $A^m = A \setminus \bigcup_{i<m} A_i$ and $B^m = B \setminus \bigcup_{i<m} B_i$. Since A^m and B^m are open we may find clopen subsets $A'_m \subset A^m$ and $B'_m \subset B^m$ such that $\mu(A'_m) > \mu(A^m)/2$ and $\mu(B'_m) > \mu(B^m)/2$. By ergodicity of T we may then choose n_m such that

$$\mu(T^{n_m} A'_m \cap B'_m) \geq \frac{1}{2}\mu(A'_m)\mu(B'_m) \geq \frac{1}{8}\mu(A^m)\mu(B^m). \tag{1}$$

We set $B_m = T^{n_m} A'_m \cap B'_m$ and $A_m = T^{-n_m} B_m$.

To see that the A_i fill out A, we must show that $\alpha := \lim \mu(A^n) = 0$. Since $\mu(A^n) = \mu(B^n) \geq \alpha$ we will have $\mu(A_{n+1}) \geq \alpha^2/8$ for all n. It follows that $\alpha = 0$ since the A_n are disjoint.

Lemma 2.

(a) If A is a clopen set in X and $\epsilon > 0$ then A can be partitioned into clopen subsets $A_1, \ldots, A_k$ such that $\mu(A_i) < \epsilon$. Consequently any open set can be written as a countable disjoint union of clopen sets with measure less than ϵ.

(b) If B is an open set in X and $\mu(B) = r_1 + r_2 + \ldots$ then there are disjoint open subsets $B_1, B_2, \ldots$ of B such that $\mu(B_i) = r_i$.

Proof: For (a) let C be any non-empty clopen set of measure less than ϵ. Suppose $A_1, A_2, \ldots, A_i$ have been defined. Let $A^i = A \backslash \bigcup_{j=1}^{i} A_i$. If $\mu(A^i) < \epsilon$ we are done. Otherwise find an n such that $\mu(T^n C \cap A^i) > \mu(C)\mu(A^i)/2$ and set $A_{i+1} = T^n C \cap A^i$. Since we always remove a fraction at least $\mu(C)/2$ from A^i we will eventually arrive at an A^k of measure less than ϵ at which point we are done.

To prove (b) we will construct for each n a sequence $B_1^n, B_2^n, \ldots$ of disjoint clopen sets such that $B_i^n = \varnothing$ for all sufficiently large i and such that for each i the sequence $B_i^1, B_i^2, \ldots$ increases to the desired B_i. Suppose that for a given n we have already found such disjoint clopen subsets $B_1^n, B_2^n, \ldots$ and that

$$r_i - \frac{1}{n} < \mu(B_i^n) < r_i.$$

By part (a) we may partition the open set $C := B \backslash \bigcup_i B_i^n$ into clopen subsets $C_1, C_2, \ldots$ of measure less than ϵ. If ϵ is sufficiently small then by taking finite unions of the sets C_i we may find disjoint clopen subsets $D_1, D_2, \ldots$ of C, all but finitely many empty, such that if we set $B_i^{n+1} = B_i^n \cup D_i$ then we have

$$r_i - \frac{1}{n+1} < \mu(B_i^{n+1}) < r_i.$$

After we have defined B_i^n for all i and n we see that the open sets $B_i = \bigcup_{n=1}^{\infty} B_i^n$ are disjoint and $\mu(B_i) = r_i$.

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Definition. *A **column** T in X consists of an ordered collection of disjoint clopen sets of the form $\{B_0, B_1, \ldots, B_{h-1}\}$ together with integers*

$$0 = n_0, n_1, \ldots, n_{h-1}$$

such that $T^{n_i} B_0 = B_i$.

We denote this column by $T = (B_i, n_i)_{i=0}^{h-1}$; h is the **height** of T and B_0 is the **base**. We will use the notation $|T| = \bigcup_i B_i$. The sets B_i are the **levels** of T. The

width of $\mathcal{T}$ is the measure of its base. If $x \in |\mathcal{T}|$ then $\mathcal{T}x$ will denote the level of $\mathcal{T}$ which contains x. If $\mathcal{T}x = B_i$ the set

$$\mathcal{O}_{\mathcal{T}}x = \{T^{n_j}(T^{-n_i}x) : j = 0, \ldots, h-1\}$$

will be called the $\mathcal{T}$- **fiber** of x. Roughly speaking what we call a column is called a tower in [HK] but we will reserve the term tower for the usual notion of a Rohlin tower, that is, a column $\mathcal{T} = (B_i, n_i)_{i=0}^{h-1}$ with $n_i = i$.

If T has no rational spectrum then the sets B_i determine the integers n_i but in general both need to be specified. Nonetheless we will sometimes refer to $\{B_0, \ldots, B_{h-1}\}$ as a column with the understanding that this means that the n_i are clear from the context, or, that we are asserting the existence of suitable n_i.

A **slice** of $\mathcal{T} = (B_i, n_i)_{i=0}^{h-1}$ is any column of the form $\mathcal{T}' = (E_i, n_i)_{i=0}^{h-1}$ such that $E_i \subset B_i$. $\mathcal{T}'$ is determined by specifying the set $E_0 \subset B_0$ and we will call $\mathcal{T}'$ the slice of $\mathcal{T}$ over E_0.

Definition. *An **array** $\mathcal{T}$ of height h is a finite or countable collection of pairwise disjoint columns $\mathcal{T}_i$ of the same height h.*

We will write $|\mathcal{T}| = \bigcup_i |\mathcal{T}_i|$. Every column is also an array. The **base** of $\mathcal{T}$ is the union of the bases of its columns and its width is the measure of its base. The **levels** of $\mathcal{T}$ are the levels of its columns and we denote the set of levels by $\mathcal{L}(T)$. (Warning: the base of $\mathcal{T}$ is not a level of $\mathcal{T}$ unless $\mathcal{T}$ has just one column.) If the array $\mathcal{T}$ is contained in a measurable subset E of X we say it is an **array partition** of E if its levels fills out E. Note that the total measure of $\mathcal{T}$ is hw where h is the height of $\mathcal{T}$ and w the width. If $\mathcal{T}$ is an array partition of X we call it simply an **array partition**.

Definition. *A **sub-array** of an array $\mathcal{T}$ is an array $\mathcal{T}'$ such that each column of $\mathcal{T}'$ is a slice of a column of $\mathcal{T}$. If $\mathcal{T}'$ fills out $\mathcal{T}$ we call it a **refinement** of $\mathcal{T}$.*

Lemma 3. *Suppose $\mathcal{T}$ is an array of width r and $r = \sum_{i=1}^{\infty} r_i$. Then $\mathcal{T}$ has disjoint subarrays $\mathcal{T}_1, \mathcal{T}_2, \ldots$ such that the width of $\mathcal{T}_i$ is r_i.*

Proof: Suppose the bases of the columns of $\mathcal{T}$ are $B_i, i \in \mathbb{N}$. The base $B = \bigcup_i B_i$ is an open set so there exist open subsets $C_1, C_2, \ldots$ of B such that $\mu(C_i) = r_i$.

Each C_i is a countable union of disjoint clopen subsets $C_{i,j}, j \geq 1$. Let $\mathcal{T}_i$ be the subarray of $\mathcal{T}$ whose columns have bases $B_k \cap C_{i,j}, j, k \geq 1$.

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Suppose $\mathcal{T}_1 = \{B_0, \ldots, B_{h-1}\}$ and $\mathcal{T}_2 = \{C_0, \ldots, C_{k-1}\}$ are disjoint columns of equal width. Suppose further that there is an integer m such that $T^m B_0 = C_0$. Then we can form the column

$$\mathcal{T} = \{B_0, \ldots, B_{h-1}, C_0, \ldots, C_{k-1}\}$$

called the **concatenation** of $\mathcal{T}_1$ and $\mathcal{T}_2$. This notion can be extended in the obvious way to define the concatenation of columns $\mathcal{T}_1, \ldots, \mathcal{T}_s$ provided their bases themselves form a column. We will denote this concatenation by $\mathcal{T} = [\mathcal{T}_1, \ldots, \mathcal{T}_s]$.

Definition. *An array $\hat{\mathcal{T}}$ is an **extension** of an array $\mathcal{T}$ if there is a refinement $\mathcal{T}'$ of $\mathcal{T}$ such that each column of $\hat{\mathcal{T}}$ is a concatenation of columns of $\mathcal{T}'$ and $\hat{\mathcal{T}}$ fills out $\mathcal{T}$. $\mathcal{T}'$ will be called the refinement of $\mathcal{T}$ associated to $\hat{\mathcal{T}}$.*

Note that $\hat{\mathcal{T}}$ and $\mathcal{T}'$ have the same set of levels. We observe that a refinement is also an extension in a trivial way. If $\hat{\mathcal{T}}$ is an extension of $\mathcal{T}$ there is a natural projection from the set of levels of $\hat{\mathcal{T}}$ to those of $\mathcal{T}$ which we denote by $\pi_{\mathcal{T}}$: $\pi_{\mathcal{T}} L = L'$ if $L \subset L'$.

Definition. *Suppose $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_{m-1}$ are disjoint arrays of equal widths. An array $\mathcal{T}$ will be called a **stacking** of $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_{m-1}$ if there are refinements $\mathcal{T}'_0, \mathcal{T}'_1, \ldots, \mathcal{T}'_{m-1}$ such that each column of $\mathcal{T}$ is a concatenation $[\mathcal{C}_0, \ldots, \mathcal{C}_{m-1}]$, where each $\mathcal{C}_i$ is a column of $\mathcal{T}'_i$, and $\mathcal{T}$ fills out $\bigcup_i |\mathcal{T}_i|$. Note that the order in which the $\mathcal{T}_i$ are listed is an essential part of this definition.*

Evidently any extension of $\mathcal{T}$ is obtained as a stacking of $\mathcal{T}_1, \ldots, \mathcal{T}_m$ where the $\mathcal{T}_i$ are disjoint subarrays of $\mathcal{T}$ which fill out $\mathcal{T}$.

Lemma 4.

(a) If $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_{m-1}$ are arrays of equal width then there is a stacking $\mathcal{T}$ of $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_{m-1}$.

(b) If $\mathcal{T}$ is an array of height h then $\mathcal{T}$ has an extension of height hm for any $m \geq 1$.

Proof:

(a) Without loss of generality we assume $m = 2$, say $\mathcal{T}_0 = \mathcal{R}$ and $\mathcal{T}_1 = \mathcal{S}$, with bases A and B. By Lemma 1 there are disjoint clopen sets $A_1, A_2, \ldots \subset A$ which fill out A and integers $n_1, n_2, \ldots$ such that the sets $B_i := T^{n_i} A_i$ are disjoint and contained in B. Let $\{C_i\}$ and $\{D_i\}$ be the bases of the columns of $\mathcal{R}$ and $\mathcal{S}$ and set

$$A_{ijk} = A_i \cap C_j \cap T^{-n_i} D_k$$

and $B_{ijk} = T^{n_i} A_{ijk}$. Then $\{A_{ijk}\}$ and $\{B_{ijk}\}$ are disjoint families of clopen subsets of A and B filling out A and B. Let $\mathcal{R}_{ijk}$ and $\mathcal{S}_{ijk}$ be the slices of $\mathcal{R}$ and $\mathcal{S}$ with bases A_{ijk} and B_{ijk}. By construction the concatenations $[\mathcal{R}_{ijk}, \mathcal{S}_{ijk}]$ exist. These concatenations are the columns of the desired stacking of $\mathcal{R}$ and $\mathcal{S}$.

(b) If $\mathcal{T}$ has width w simply divide $\mathcal{T}$ into subarrays $\mathcal{T}_i$, $i = 0, \ldots, m-1$ of width w/m and then use part (a) to stack these.

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We define the **diameter** of an array $\mathcal{T}$, denoted diam $\mathcal{T}$, as the supremum of the diameters of its levels.

Lemma 5. *Any array $\mathcal{T}$ has a refinement $\mathcal{T}'$ with* diam $\mathcal{T}' < \epsilon$.

Proof: Without loss of generality $\mathcal{T}$ has just one column $(B_i, n_i)_{i=0}^{h-1}$. Using the fact that X is fractured it is easy to see that there is a countable partition P of X into clopen sets of diameter less than ϵ. For $p = (p_0, \ldots, p_{h-1}) \in P^h$ let

$$B_p = \bigcap_{i=0}^{h-1} T^{-n_i}(p_i \cap B_i).$$

Then the clopen sets B_p, $p \in P^h$ are disjoint and cover B_0. The slices of $\mathcal{T}$ over the sets $B_p, p \in P^h$ form the desired refinement. ∎

The following lemma is just the usual Rohlin lemma but with a tower whose levels are clopen.

Lemma 6. *For every $h \in \mathbb{N}$ and $\epsilon > 0$ there is a clopen set B such that the sets $B, TB, \ldots, T^{h-1}B$ are disjoint and cover at least $1 - \epsilon$ of X.*

Proof: Just repeat the usual proof of the Rohlin Lemma (see for example [F]) starting from a small clopen set. Alternately one can take a measurable Rohlin tower, approximate its base B by a clopen set C and then disjointify the images of C.

<div style="text-align:right">■</div>

The following lemma is key to the proof of Theorem 2. It says roughly that any array partition has an extension whose fibers fill out large segments of the orbits of T.

Lemma 7. *Suppose $\mathcal{T}$ is an array partition of X and $\epsilon > 0$. Then one can find an extension $\hat{\mathcal{T}}$ of $\mathcal{T}$ and a clopen set $X^\sharp$ such that $X^\sharp \subset |\hat{\mathcal{T}}|$, $\mu(X^\sharp) > 1 - \epsilon$ and for all $x \in X^\sharp$ we have $Tx \in \mathcal{O}_{\hat{\mathcal{T}}}x$.*

Proof: Given a column $\mathcal{C} = (C_i, n_i)_{i=0}^{h-1}$ we will refer to the integer $n = \max_{i,j}|n_i - n_j|$ as the **spread** of $\mathcal{T}$. Suppose $\mathcal{T}$ has height s and columns $\mathcal{T}_i, i = 1, 2, \ldots$, and let $\mathcal{L} = \mathcal{L}(T)$ denote the collection of levels of $\mathcal{T}$. Fix a $\delta > 0$ to be specified in the course of the proof and take a finite collection $\{\mathcal{T}_i, i = 1, \ldots, k\}$ of columns of $\mathcal{T}$ which covers $1 - \delta$ of X and agree to call these the **good** columns. Let $X' = \bigcup_{i=1}^{k}|\mathcal{T}_i|$. Let m be the maximum of the spreads of the good columns.

Find a clopen Rohlin tower for T with base E and height $h \gg m$, to be further specified later, which covers more than $1 - \delta$ of X. For each $L = (L_0, L_1, \ldots, L_{h-1}) \in \mathcal{L}^h$ let $E_L = \bigcap_j T^{-j}(T^j E \cap L_j)$. Then the sets $E_L, L \in \mathcal{L}^h$ are disjoint clopen subsets of E which fill out E. Each E_L is the base of a Rohlin tower of height h and $T^j E_L \subset L_j$ for $0 \leq j \leq h - 1$.

Let us call the union of the top and bottom m levels of the tower the **buffer**. We assume that h has been chosen large enough so that the buffer has measure less than δ. Fixing for the moment any $L \in \mathcal{L}^h$ and any j such that $m < j < h - m$,

let $(C_i, n_i)_{i=0}^{s-1}$ (temporarily) denote the column of $\mathcal{T}$ which contains $T^j E_L$. If $T^j E_L \subset C_r$ then we will denote the slice of $(C_i, n_i)_{i=0}^{s-1}$ over $T^{-n_r} T^j E_L$ by $\mathcal{C}_{L,j}$, so that $T^j E_L$ is the r-th level of $\mathcal{C}_{L,j}$. We will be interested in those pairs (L, j), call them **good**, such that $m < j < h - m$ and $T^j E_L$ is contained in a level of one of the good columns $\mathcal{T}_i, 1 \leq i \leq k$. Note that for a good (L, j) the levels of $\mathcal{C}_{L,j}$ are all levels of the T-tower of height h over E_L, since they are of the form $T^s T^j E_L$ with $|s| \leq m$ and $m < j < h - m$. Moreover, for a fixed $L \in \mathcal{L}^h$ and any two good (L, j) and (L, j'), $\mathcal{C}_{L,j}$ and $\mathcal{C}_{L,j'}$ are either disjoint or equal. This is clear: $\mathcal{C}_{L,j}$ and $\mathcal{C}_{L,j'}$ are both slices of $\mathcal{T}$ and each consists of levels of the T-tower of height h over E_L so if they are not disjoint they share a level. Clearly if two slices of $\mathcal{T}$ share a level they are identical.

Let $\mathcal{S}_L$ denote the set of distinct slices $\mathcal{C}_{L,j}, m < j < h - m$ such that (L, j) is good. Since $\mu(X') > 1 - \delta$ and the measure of the buffer is less than δ, a 'Fubini' argument shows that E is $(1 - \delta_1)$-covered by E_L's for which the slices in $\mathcal{S}_L$ cover at least a fraction $1 - \delta_1$ of $\bigcup_j T^j E_L$, where $\delta_1 = \sqrt{2\delta}$. Call these the good L's and denote the set of good L's by $\mathcal{G}$. Let r be the least integer such that $rs \geq (1 - \delta_1)h$. Then for each good L the cardinality of $\mathcal{S}_L$ is at least r so we may choose r elements of $\mathcal{S}_L$ and concatenate them in any order to form a column $\mathcal{C}_L$ of height rs. The concatenation is possible because the bases of these r columns themselves form a column, as they are all levels of the tower of height h over E_L. Our choice of r implies that if L is good then $\mathcal{C}_L$ covers at least a fraction $1 - \delta_1$ of $\bigcup_{j=0}^{h-1} T^j E_L$. Thus we have

$$\mu\left(\bigcup\{|\mathcal{C}_L| : L \in \mathcal{G}\}\right) > (1 - \delta_1)^2(1 - \delta) =: 1 - \delta_2.$$

Choose a finite subset $\mathcal{G}' \subset \mathcal{G}$ such that the measure of the clopen set

$$X^\flat = \cup\{|\mathcal{C}_L| : L \in \mathcal{G}'\})$$

is greater then $1 - 2\delta_2 =: 1 - \delta_3$. The $\mathcal{C}_L, L \in \mathcal{G}'$, form an array $\mathcal{T}_0$ contained in X' with finitely many columns which will be part of the desired extension $\hat{\mathcal{T}}$. Since $|\mathcal{T}_0|$ is clopen what remains of $\mathcal{T}$ is again an array $\mathcal{T}_1$. It remains only to form any extension $\mathcal{T}_2$ of $\mathcal{T}_1$ of height rs and adjoin it to $\mathcal{T}_0$ to obtain $\hat{\mathcal{T}}$.

Now set $X^\sharp = X^\flat \cap T^{-1} X^\flat \cap (T^{h-1} E)^c$ so

$$\mu(X^\sharp) > 1 - 2\delta_3 - \frac{1}{h} > 1 - \epsilon,$$

if δ is sufficiently small and h sufficiently large. If $x \in X^\sharp$ then x and Tx belong to the same fiber of the T-tower of height h over E, since $x \notin T^{h-1} E$. Since both x and Tx belong to $X^\flat$ they are in the same fiber of $\mathcal{C}_L$ for some $L \in \mathcal{G}$, hence x and Tx are in the same fiber of $\hat{T}$.

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Definition. *Suppose that $\mathcal{T}$ and $\mathcal{S}$ are array partitions of X and Y of the same height. An **array map** from $\mathcal{T}$ to $\mathcal{S}$ is a map from the set of levels of $\mathcal{T}$ to those of $\mathcal{S}$ which maps individual columns bijectively to individual columns, preserves the order on each column and maps μ to ν: if $L \in \mathcal{L}(S)$ is a level of $\mathcal{S}$ then $\mu(\bigcup \varphi^{-1}\{L\}) = \nu(L)$.*

To clarify the notation, the expression $\bigcup \varphi^{-1}\{L\}$ can be read literally but it could also be written as $\bigcup\{L' \in \mathcal{L}(\mathcal{T}) : \varphi(L') = L\}$. The map φ can also be viewed as a mapping from the set of columns of $\mathcal{T}$ to the set of columns of $\mathcal{S}$. One may visualize φ as lumping together a sub-array of $\mathcal{T}$ (whose columns are columns of $\mathcal{T}$ rather than slices) with total width w to give one column of $\mathcal{S}$ of width w. We want to stress however that φ is a mapping of sets, not points.

Suppose $\varphi : \mathcal{T} \to \mathcal{S}$ is an array map and suppose $\hat{\mathcal{T}}$ and $\hat{\mathcal{S}}$ are extensions of $\mathcal{T}$ and $\mathcal{S}$ with associated refinements $\mathcal{T}'$ and $\mathcal{S}'$. We will use $\pi_{\mathcal{T}}$ to denote the projection from $\hat{\mathcal{T}}$ to $\mathcal{T}$ and $\pi_{\mathcal{S}}$ for the projection from $\hat{\mathcal{S}}$ to $\mathcal{S}$. An array map $\psi : \hat{\mathcal{S}} \to \hat{\mathcal{T}}$ will be called an **extension** of φ if $\varphi \circ \pi_{\mathcal{T}} \circ \psi = \pi_{\mathcal{S}}$. The fact that ψ preserves the order on columns implies that ψ is also an array map from $\mathcal{S}'$ to $\mathcal{T}'$. Similarly if $\psi : \hat{\mathcal{T}} \to \hat{\mathcal{S}}$ we call it an extension of φ if $\pi_{\mathcal{S}} \circ \psi = \varphi \circ \pi_{\mathcal{T}}$ and again ψ is automatically also an array map from $\mathcal{T}'$ to $\mathcal{S}'$. Analogous definitions hold if $\varphi : \mathcal{S} \to \mathcal{T}$. By chasing commutative diagrams it is easy to see that if $\mathcal{T}''$ and $\mathcal{S}''$ are extensions of $\mathcal{T}'$ and $\mathcal{S}'$ and $\chi : \mathcal{T}'' \to \mathcal{S}''$ is an extension of ψ then it is an extension of φ, whatever the directions of the maps φ, ψ and χ.

Lemma 8 (Copying Extensions). *Suppose $\varphi : \mathcal{T} \to \mathcal{S}$ is an array map and suppose $\hat{\mathcal{T}}$ is an extension of $\mathcal{T}$. Then there is an extension $\hat{\mathcal{S}}$ of S and an array map $\psi : \hat{\mathcal{S}} \to \hat{\mathcal{T}}$ which extends φ.*

Proof: Suppose the heights of $\mathcal{T}$ and $\hat{\mathcal{T}}$ are h and mh. Let $\mathcal{T}'$ denote the refinement of $\mathcal{T}$ associated to $\hat{\mathcal{T}}$. Let $\{\mathcal{S}_i : i \in I\}$ denote the set of columns of $\mathcal{S}$. Label each column of $\mathcal{T}$ by $i \in I$ if it is mapped to $\mathcal{S}_i$. For each column $\mathcal{C} \in \hat{\mathcal{T}}$ let $w_{\mathcal{C}}$ denote its width and $i_{\mathcal{C}} = (i_{\mathcal{C},0}, \ldots, i_{\mathcal{C},m-1})$ its string of labels: $\mathcal{C}$ is a concatenation $[\mathcal{C}_0, \ldots, \mathcal{C}_{m-1}]$ where each $\mathcal{C}_j$ is a slice of a column of $\mathcal{T}$ which has label $i_{\mathcal{C},j}$.

Now find a system of disjoint subarrays

$$\{\mathcal{S}_{\mathcal{C},j}, \mathcal{C} \in \hat{\mathcal{T}}, 0 \leq j < m\}$$

of $\mathcal{S}$ such that $\mathcal{S}_{\mathcal{C},j}$ is a subarray of $\mathcal{S}_{i_{\mathcal{C},j}}$ and has width $w_{\mathcal{C}}$. This is possible because of Lemma 3 and the measure-preserving character of φ. Observe that the $\mathcal{S}_{\mathcal{C},j}$ fill out $\mathcal{S}$. Now use Lemma 4 to find a stacking $\mathcal{S}_{\mathcal{C}}$ of $\mathcal{S}_{\mathcal{C},0}, \mathcal{S}_{\mathcal{C},2}, \ldots \mathcal{S}_{\mathcal{C},m-1}$. The $\mathcal{S}_{\mathcal{C}}$, $\mathcal{C} \in \hat{\mathcal{T}}$, together form the array partition $\hat{\mathcal{S}}$ we seek. Now define ψ by mapping each column of $S_{\mathcal{C}}$ to $\mathcal{C}$.

$\blacksquare$

We are now ready to prove Theorem 2. Using Lemmas 5,7 and 8 we inductively construct a sequence of arrays $\mathcal{T}_i, i = 0, 1, \ldots$ in X, a sequence of arrays $\mathcal{S}_i, i = 0, 1, \ldots$ in Y and array maps $\varphi_i : \mathcal{T}_i \to \mathcal{S}_i$ for even i and $\psi_i : \mathcal{S}_i \to \mathcal{T}_i$ for odd i such that, for each i, $\mathcal{T}_{i+1}$ extends $\mathcal{T}_i$ and $\mathcal{S}_{i+1}$ extends $\mathcal{S}_i$ and

  (i) for each even i, φ_i extends ψ_{i-1} and ψ_{i+1} extends φ_i.

  (ii) diam $\mathcal{T}_i \to 0$ and diam $\mathcal{S}_i \to 0$

  (iii) for each even $i > 0$ there is a clopen set $X_i \subset \mathcal{T}_i$ such that $\mu(X_i) > 1 - \frac{1}{i}$ and for each $x \in X_i$ we have $Tx \in \mathcal{T}_i x$; for each even i there is a clopen $Y_i \subset \mathcal{S}_i$ satisfying the analogous conditions.

We remark that (i) implies that any φ_i or ψ_i is an extension of any φ_j or ψ_j for $j < i$. Let $X' = \bigcap_i |\mathcal{T}_i|$ and $Y' = \bigcap_i |\mathcal{S}_i|$. These are G_δ subsets of full measure,

13

not necessarily invariant. We define a map $\varphi : X' \to Y'$ as follows. For $x \in X'$, $\{\mathcal{T}_{2i}x\}$ is a decreasing sequence of clopen sets whose intersection is $\{x\}$. Since φ_{2i+2} extends φ_{2i} it follows that $\varphi_{2i}(\mathcal{T}_{2i}x)$ is also a decreasing sequence of clopen sets. Since the diameters go to 0 it follows that the intersection is a singleton $\{y\}$ such that $y \in Y'$ and we define $\varphi(x) = y$. We define $\psi : Y' \to X'$ similarly using the maps ψ_{2i+1}. Note that the definition of φ boils down to $\mathcal{S}_{2i}\varphi(x) = \varphi_{2i}(\mathcal{T}_{2i}x)$.

Let $\mathcal{L} = \bigcup_i \mathcal{L}(\mathcal{S}_{2i})$. The collection $\{L \cap Y' : L \in \mathcal{L}\}$ generates the topology of Y', since its members are open and $\mathrm{diam}\mathcal{S}_{2i} \to 0$. Thus to show φ is continuous it suffices to show that for all i and all $L \in \mathcal{S}_{2i}$, we have $\varphi^{-1}(L \cap Y')$ open in the relative topology of X'. But it is clear that

$$\varphi^{-1}(L \cap Y') = (\bigcup \varphi_{2i}^{-1}\{L\}) \cap X' \ :$$

evidently

$$\varphi((\bigcup \varphi_i^{-1}\{L\}) \cap X') \subset L \cap Y'$$

and on the other hand if $x \in X'$ and $\mathcal{T}_{2i}x$ is disjoint from $\bigcup \varphi_{2i}^{-1}\{L\} \cap X'$ then φx is in $\varphi_{2i}(\mathcal{T}_{2i}x)$ which is disjoint from L. Thus $\varphi^{-1}(L \cap Y')$ is open in X', we have shown that φ is continuous, and the same argument holds for ψ.

Next we check that φ and ψ are inverse maps. For $x \in X'$ we have

$$
\begin{aligned}
\mathcal{T}_{2i-1}\psi\varphi x &= \psi_{2i-1}\mathcal{S}_{2i-1}\varphi x \\
&= \psi_{2i-1}\pi_{\mathcal{S}_{2i-1}}\mathcal{S}_{2i}\varphi x \\
&= \psi_{2i-1}\pi_{\mathcal{S}_{2i-1}}\varphi_{2i}\mathcal{T}_{2i}x \\
&= \pi_{\mathcal{T}_{2i-1}}\mathcal{T}_{2i}x \\
&= \mathcal{T}_{2i-1}x.
\end{aligned}
$$

Since $\mathrm{diam}\mathcal{T}_{2i-1}x \to 0$ it follows that $\psi\varphi x = x$. Similarly $\varphi \circ \psi = \mathrm{id}_{Y'}$ so ψ is a homeomorphism from X' onto Y' and $\psi = \varphi^{-1}$.

Now we claim that for $x \in X'$ we have $\varphi(\mathcal{O}_{\mathcal{T}_i}x) = \mathcal{O}_{\mathcal{S}_i}\varphi(x)$ for each even i. (Note that if $x \in X'$ then $\mathcal{O}_{\mathcal{T}_i}x \subset X'$ for each i). For any even $j > i$ let $\mathcal{T}_{ij}$ denote the refinement of $\mathcal{T}_i$ determined by its extension $\mathcal{T}_j$ and $\mathcal{C}_{ij}x$ the column of $\mathcal{T}_{ij}$

14

containing x, with analogous definitions in Y. Clearly $\bigcap_j |\mathcal{C}_{ij}x| = \mathcal{O}_{\mathcal{T}_i}x$. Since φ_j is an array map from $\mathcal{T}_{ij}$ to $\mathcal{S}_{ij}$ it is clear that $\varphi(|\mathcal{C}_{ij}x| \cap X') \subset |\mathcal{C}_{ij}\varphi(x)|$. Thus

$$\varphi(\mathcal{O}_{\mathcal{T}_i}x) = \varphi\Big(\bigcap_j |\mathcal{C}_{ij}x|\Big) = \bigcap_j \varphi(|\mathcal{C}_{ij}x|) \subset \bigcap_j |\mathcal{C}_{ij}\varphi(x)| = \mathcal{O}_{\mathcal{S}_i}(\varphi x).$$

Since φ is injective we conclude that $\varphi(\mathcal{O}_{\mathcal{T}_i}x) = \mathcal{O}_{\mathcal{S}_i}x$.

Recall that X_n, as defined in (iii) above, is clopen. Let $X^* = (\bigcup_i X_{2i}) \cap X' \cap T^{-1}X'$, so X^* is a G_δ of full measure. We claim that for $x \in X^*$ we have $\varphi(Tx) \in \mathcal{O}_S \varphi(x)$. This is clear: if $x \in X^*$, so $x \in X_{2i}$ for some i, then $Tx \in \mathcal{O}_{\mathcal{T}_{2i}}x$ so

$$\varphi(Tx) \in \varphi(\mathcal{O}_{\mathcal{T}_{2i}}x) = \mathcal{O}_{\mathcal{S}_{2i}}\varphi x \subset \mathcal{O}_S \varphi x$$

as we claimed. Similarly we obtain a dense G_δ of full measure $Y^* \subset Y' \cap S^{-1}Y'$ such that for $y \in Y^*$ we have $\psi(Sy) \in \mathcal{O}_T \psi y$.

Now X^* and Y^* need not be invariant nor do we have $\varphi(X^*) = Y^*$. We remedy this as follows. Given a G_δ subset E of full measure in X we will write E^T for the set $\bigcap_{n \in \mathbb{Z}} T^n E$, which is an invariant G_δ subset of E of full measure. We will use the same notation in Y. Let

$$X_1 = (X^* \cap \psi Y^*)^T,$$

$$Y_1 = (\varphi(X_1))^S,$$

$$X_2 = (\psi(Y_1))^T,$$

$$Y_2 = (\varphi(X_2))^S$$

and so on. Let $X_\infty = \bigcap_n X_n$ and $Y_\infty = \bigcap_n Y_n$. Then X_∞ and Y_∞ are invariant G_δ subsets of X^* and Y^*, both having full measure, and evidently $\varphi X_\infty = Y_\infty$. For $x \in X_\infty$ $\varphi(Tx) \in \mathcal{O}_S \varphi(x)$ since $X_\infty \subset X^*$ and because X_∞ is invariant it follows easily that $\varphi \mathcal{O}_T^+ x \subset \mathcal{O}_S \varphi(x)$ for all $x \in X_\infty$, where $\mathcal{O}^+$ denotes the forward orbit. Now since we also have

$$\varphi(\mathcal{O}_T^+ x) \subset \varphi(\mathcal{O}_T^+ T^{-1}x) \subset \mathcal{O}_S \varphi(T^{-1}x)$$

15

it follows that $\mathcal{O}_S\varphi(x) = \mathcal{O}_S\varphi(T^{-1}x)$ and in particular $\varphi(T^{-1}x) \in \mathcal{O}_S\varphi(x)$. In a similar way we find that $\varphi(T^{-i}x) \in \mathcal{O}_S\varphi(x)$ for all $i > 0$ so we conclude that $\varphi(\mathcal{O}_T x) \subset \mathcal{O}_S\varphi(x)$ for any $x \in X_\infty$. All we used to show this is that X_∞ is invariant and contained in X^*. It follows that the corresponding fact for ψ holds as well, even though the definitions of X_∞ and Y_∞ are not symmetric. Using this we see that

$$\mathcal{O}_T x = \psi\varphi\mathcal{O}_T x \subset \psi\mathcal{O}_S\varphi(x) \subset \mathcal{O}_T\psi(\varphi(x)) = \mathcal{O}_T x.$$

It follows that all the containments are in fact equalities and in particular that $\varphi(\mathcal{O}_T x) = \mathcal{O}_S\phi(x)$. This concludes the proof of Theorem 2.

Section 3. The II$_\infty$ case.

In this section we shall prove the following result by reducing it to Theorem 1.

Theorem 3. *Suppose X and Y are Polish spaces with non-atomic infinite σ-finite measures μ and ν. Suppose that X and Y have at least one open subset of finite non-zero measure. Suppose that T and S are ergodic measure-preserving homeomorphisms of (X, μ) and (Y, ν). Then there are invariant G_δ subsets $X' \subset X$ and $Y' \subset Y$ of full measure and a homeomorphism $\varphi : X' \to Y'$ which maps $\mu|_{X'}$ to $\nu|_{Y'}$ and maps T-orbits onto S-orbits.*

Proof: As in the measurable case, we shall prove Theorem 3 by inducing on a set of finite measure. We begin with some remarks about inducing in our setting. First observe that the hypotheses of Theorem 3 imply that T is conservative and that for any non-null set $E \subset X$ the set $\bigcup_{i>0} T^i E$ is co-null. If U is a non-empty clopen subset of X then it is easy to see that the induced transformation T_U is a homeomorphism from an open subset U' of U, such that $\mu(U \backslash U') = 0$, to another open subset of U. If we assume further that for all $x \in X$ the T-orbit of x intersects U infinitely often in both positive and negative time then T will be a homeomorphism from U to itself. Call such sets U T-**good**. If U is not T-good, observe that the set X' of points in X whose T-orbit intersects U infinitely often in both positive and negative time is an invariant co-null G_δ. Setting $U' = U \cap X'$ then U' is clopen in X' and it is $T_{X'}$-good. (The induced map $T_{X'}$ is the same as

16

the restriction of T to X'). This means that for the purpose of proving Theorem 3 there is no loss of generality in assuming that U itself is T-good. Note also that if U is T-good and X' is any invariant co-null G_δ then $U' = U \cap X'$ is $T_{X'}$-good. Finally, any clopen superset of a T-good set is T-good.

To prove Theorem 3 we may assume, as in the proof of Theorem 1, that X and Y are fractured and that μ and ν have full support. Let U be a non-empty clopen set of finite measure in X. Then $\bigcup_{i\geq0} T^i U$ is co-null G_δ and after restricting to $(\bigcup_{i\geq0} T^i U)^T$ we might as well assume that $\bigcup_{i\geq0} T^i U = X$. It follows easily that X can also be expressed as a countable disjoint union of of clopen sets U_i of finite measure. By the above remarks there is no loss of generality in assuming that each U_i is T-good. Applying Lemma 2(b) to the induced system $(U_i, \mu|_{U_i}, T_{U_i})$ we see that U_i contains disjoint open subsets $U_{i,j}$ of any desired measures which sum to $\mu(U_i)$. From this it follows that we can find disjoint open subsets $E_0, E_1, \ldots$ of X such that $\mu(E_i) = 1$ for all i and $X' := \bigcup_{i\geq0} E_i$ is co-null. Restricting to X' there is no loss of generality in assuming that $\bigcup_{i\geq0} E_i = X$. In addition we may assume that E_0 is T-good. Since $E_0 \cup E_i$ is also T-good we may apply Lemma 1 to $T_{E_0 \cup E_i}$ to find open subsets E_0' and E_i' and a homeomorphism θ_i from E_0' onto E_i' such that θ_i is a piecewise power of $T_{E_0 \cup E_i}$, and hence also a piecewise power of T. Cutting down to $(\bigcup E_i)^T$ we may assume that θ_i maps all of E_0 to all of E_i. It is easy to see that $\theta_i(\mathcal{O}_T x \cap E_0) = \mathcal{O}_T x \cap E_i$ for any $x \in X$. Letting $T_0 = T_{E_0}$ we observe also that for $x \in E_0$ we have $\mathcal{O}_{T_0} x = \mathcal{O}_T x \cap E_0$.

Now go through the same process in Y to obtain corresponding sets F_i and homeomorphisms ψ_i. Let $T_0 = T_{E_0}$, $S_0 = S_{F_0}$, $\mu_0 = \mu|_{E_0}$ and $\nu_0 = \nu|_{F_0}$. Applying Theorem 1 to the homeomorphisms T_0 and S_0 with measures μ_0 and ν_0 we find invariant G_δ subsets $E_0' \subset E_0$ and $F_0' \subset F_0$, both of measure one and a homeomorphism $\varphi : E_0' \to F_0'$ which maps T_0-orbits onto S_0-orbits and μ_0 to ν_0. Now set $E_i' = \theta_i E_0'$, $X' = \bigcup E_i'$, $F_i' = \psi_i F_0'$ and $Y' = \bigcup F_i'$. Then X' and Y' are invariant, co-null G_δ subsets of X and Y. Define a homeomorphism $\Phi : X' \to Y'$ by

$$\Phi(x) = \psi_i \varphi \theta_i^{-1} x \text{ for } x \in E_i'.$$

Then it is clear that Φ is the desired orbit equivalence. ∎

17

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